CR.1

Cauchy-Riemann Equations – necessary (but not sufficient) condition for existence of $f'(z_0)$ at $z = z_0$.

Consider $$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z}$$

by writing $f(z) = u(x,y) + i\,v(x,y)$

Letting $z_0 = x_0 + iy_0$, $\Delta z = \Delta x + i\Delta y$

then
$$\mathrm{Re}[f'(z_0)] = \lim_{(\Delta x, \Delta y) \to (0,0)} \mathrm{Re}\left[\frac{f(z_0+\Delta z) - f(z_0)}{\Delta z}\right]$$

and
$$\mathrm{Im}[f'(z_0)] = \lim_{(\Delta x, \Delta y) \to (0,0)} \mathrm{Im}\left[\frac{f(z+\Delta z) - f(z_0)}{\Delta z}\right]$$

– limit of sum = sum of limits

$$\frac{f(z_0+\Delta z) - f(z_0)}{\Delta z} = \frac{u(x_0+\Delta x, y_0+\Delta y) - u(x_0,y_0) + i\left[v(x_0+\Delta x, y_0+\Delta y) - v(x_0,y_0)\right]}{\Delta x + i\Delta y}$$

– these expressions are valid irrespective of mode of approach to origin.

CR.2

Lets let $(\Delta x, \Delta y) \to (0,0)$ horizontally along real axes

i.e $\Delta y = 0$

$$\therefore \operatorname{Re}[f'(z_0)] = \lim_{\Delta x \to 0} \frac{u(x_0+\Delta x, y_0) - u(x_0, y_0)}{\Delta x} = \frac{\partial u}{\partial x}(x_0, y_0)$$

$$\operatorname{Im}[f'(z_0)] = \lim_{\Delta x \to 0} \frac{v(x_0+\Delta x, y_0) - v(x_0, y_0)}{\Delta x} = \frac{\partial v}{\partial x}(x_0, y_0)$$

$$\therefore f'(z_0) = u_x(x_0, y_0) + i\, v_x(x_0, y_0)$$

Now ~~approach~~ let $(\Delta x, \Delta y) \to (0,0)$ vertically along the imaginary axes i.e $\Delta x = 0$

$$\operatorname{Im}[f'(z_0)] = \lim_{\Delta y \to 0} \frac{u(x_0, y_0+\Delta y) - u(x_0, y_0)}{i\, \Delta y} = -i \frac{\partial u}{\partial y}(x_0, y_0)$$

$$\operatorname{Re}[f'(z_0)] = \lim_{\Delta y \to 0} \frac{v(x_0, y_0+\Delta y) - v(x_0, y_0)}{\Delta y} = \frac{\partial v}{\partial y}(x_0, y_0)$$

$$\therefore f'(z_0) = v_y(x_0, y_0) - i\, u_y(x_0, y_0)$$

$$= -i\left[u_y(x_0, y_0) + i\, v_y(x_0, y_0)\right]$$

CR.3

But $f'(z_0)$ must be same irrespective of mode of approach of $(\Delta x, \Delta y) \to (0,0)$

$$\therefore \quad u_x(x_0,y_0) + i v_x(x_0,y_0) = v_y(x_0,y_0) - i u_y(x_0,y_0)$$

Equating real and imaginary parts separately

$$\boxed{\begin{aligned} u_x(x_0,y_0) &= v_y(x_0,y_0) \\ u_y(x_0,y_0) &= -v_x(x_0,y_0) \end{aligned}}$$

- Cauchy - Riemann equations are pde's which must be satisfied for $f'(z_0)$ to exist.

N.B. Cauchy - Riemann equations were derived by approaching $(0,0)$ along two special directions i.e real and imaginary axis. As existence of $f'(z_0)$ requires independence of approach in any direction i.e for $\lim_{\Delta z \to 0} \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z}$ to exist, C.R equations are necessary but not sufficient [Prob 5 HW].